19 More on Eigenvalue Problems

Below we touch on a sometimes useful monotonicity theorem for comparing sets of eigenvalues, then move on to a brief discussion of some eigenvalue problem topics in higher dimensions.

19.1 A Monotonicity Theorem

Consider the Sturm-Liouville EVP

$$\begin{cases} \frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) - q(x)\phi + \lambda\sigma(x)\phi = 0 \quad \text{in } a < x < b \\ \\ \phi(a) = \phi(b) = 0 \end{cases} \tag{1}$$

That is, p, q, σ satisfy the conditions in the Sturm-Liouville theorem of the last section.

Theorem: Reducing the interval (a, b), or increasing p, or increasing q, or decreasing σ, increases all eigenvalues of problem (1).

This means, for example, if (1) has real, ordered, positive eigenvalues $\{\lambda_n\}_{n\geq 1}$ and we have another problem that is identical to (1) except we change $p(x)$ to $\tilde{p}(x)$ such that $\tilde{p}(x) \geq p(x)$ on (a, b), then the new problem has eigenvalues $\{\tilde{\lambda}_n\}_{n\geq 1}$, and by the statement of this theorem, $\tilde{\lambda}_n \geq \lambda_n$ for each $n = 1, 2, \ldots$. A way of remembering this theorem is to consider the problem with all constant coefficients so we have an explicit expression for the eigenvalues. Consider (1) with positive, constant coefficients. Then

$$p\frac{d^2\phi}{dx^2} - q\phi + \lambda\sigma\phi = 0$$

which can be written as

$$\frac{d^2\phi}{dx^2} + \frac{\lambda\sigma - q}{p}\phi = \frac{d^2\phi}{dx^2} + \mu\phi = 0 \ .$$

Hence, $\phi(x) = \sin(\sqrt{\mu}(x-a))$ satisfies the equation and the left-hand boundary condition. With $\sin(\sqrt{\mu}(b-a)) = 0$, we have $\sqrt{\mu}(b-a) = n\pi$, $n = 1, 2, \ldots$. Therefore, $\mu = \mu_n = \left(\frac{n\pi}{b-a}\right)^2$, or

$$\lambda_n = \frac{q}{\sigma} + \frac{p}{\sigma}\left(\frac{n\pi}{b-a}\right)^2 \quad n = 1, 2, \ldots \ . \tag{2}$$

Now we can see that λ_n can be increased by increasing p or q, or decreasing σ or $b-a$. The above monotonicity theorem just states this also holds when one or more of the coefficients are x-dependent.

Example: Consider the problem

$$\begin{cases} \frac{d^2\phi}{dx^2} - q(x)\phi + \lambda\phi = 0 \\ \\ \phi(0) = \phi(1) = 0 \end{cases}$$

where $q(x)$ is a continuous function on $[0,1]$. What can we say about the eigenvalues (other than what is mentioned in the Sturm-Liouville theory)?

Well, from (2), if $q(x)$ is a constant $\bar{q} \geq 0$, then $\lambda_n = \bar{q} + (n\pi)^2$ (since $\sigma = p \equiv 1$, $b-a = 1$). In general $q = q(x)$ is non-constant, so we can not obtain an explicit formula for $\lambda_n = \lambda_n(q)$, $n = 1, 2, \ldots$. but q being continuous on the closed interval means it has a maximum value, q_M, and a minimum value, q_m, on the interval $[0,1]$, so $q_m \leq q(x) \leq q_M$. Invoking the monotonicity theorem (twice), we have

$$q_m + n^2\pi^2 \leq \lambda_n(q) \leq q_M + n^2\pi^2 \qquad \text{for each } n.$$

In particular, $\lim_{n\to\infty} \frac{\lambda_n(q)}{n^2\pi^2} = 1$. That is, $\lambda_n(q)$, no matter what q is, looks more like $n^2\pi^2$ as n gets larger. (Mathematicians like to write this statement as $\lambda_n(q) \sim n^2\pi^2$ as $n \to \infty$.)

Remark: This example is a pretty prominent EVP, so it has been investigated enough that we have more information about its eigenvalues. As long as $q(x)$ is square integrable (i.e. $\int_0^1 (q(x))^2 \, dx < \infty$), it need not be continuous everywhere),

$$\lambda_n(q) = n^2\pi^2 + \int_0^1 q(y) \, dy - \int_0^1 q(y)\cos(2n\pi y) \, dy + \epsilon_n$$

where the error ϵ_n satisfies $|\epsilon_n| \leq C/n$, for some constant C independent of n, or q. The last integral goes to zero by the Riemann-Lebesque lemma, so $\lambda_n(q) = n^2\pi^2 + \{\text{average of } q \text{ on } [0,1]\} + \text{the error}$,
where the error goes to zero like $1/n$ as $n \to \infty$. This is a nice correction to the approximation $\lambda_n(q) \sim n^2\pi^2$.

Example: From our previously introduced variable-density vibrating string problem, on page 2 of the previous section, consider

$$\begin{cases} \frac{d^2\phi}{dx^2} + \lambda(1+x)^{-2}\phi = 0\ , \quad 0 < x < l \\ \\ \phi(0) = \phi(l) = 0 \end{cases}$$

Note that on $(0, l)$, $(1+l)^{-2} \leq \sigma(x) = (1+x)^{-2} \leq 1$. If we consider instead the problem

$$\begin{cases} \frac{d^2\phi}{dx^2} + \mu(1+l)^{-2}\phi = 0\ , \quad 0 < x < l \\ \\ \phi(0) = \phi(l) = 0 \end{cases}$$

we find the eigenvalues to be $\mu = \mu_n = \left(\frac{n\pi(1+l)}{l}\right)^2$, $n = 1, 2, \ldots$. By the monotonicity theorem we have replaced $\sigma(x)$ by something smaller, so $\mu_n \geq \lambda_n$, $n \geq 1$. Similarly, if we replace $\sigma(x)$ with 1,

$$\begin{cases} \frac{d^2\phi}{dx^2} + \mu\phi = 0 \qquad 0 < x < l \\ \\ \phi(0) = \phi(l) = 0 \end{cases}$$

then $\mu_n = \left(\frac{n\pi}{l}\right)^2$. Therefore, by the monotonicity theorem, we have the eigenvalue bounds

$$\left(\frac{n\pi}{l}\right)^2 \leq \lambda_n \leq \left(\frac{n\pi}{l}\right)^2 (1+l)^2 \quad n = 1, 2, \ldots$$

Exercise: Find upper and lower bounds on the nth eigenvalue λ_n for the EVP

$$\begin{cases} \frac{d}{dx}\left((1+x^2)\frac{d\phi}{dx}\right) - x\phi + \lambda(1+x^2)\phi = 0 \quad 0 < x < 1 \\ \\ \phi(0) = 0 = \phi(1) \end{cases}$$

Before leaving the discussion of EVPs, let us consider some analogous results to previously mentioned properties that can arise in a multi-space dimensional setting. We will leave a fuller discussion of solving such equations for later.

19.2 Some Remarks on Eigenvalue Problems in Higher Dimensions

For general consideration below, let Ω be a bounded, simply-connected region in $\mathbb{R}^N$ $(N \geq 2)$, with smooth boundary $\partial\Omega$; that is, the unit outward-pointing normal vector $\nu = \nu(\mathbf{x})$ is defined everywhere on $\partial\Omega$.

Example: If we have the diffusion equation

$$u_t = D\nabla^2 u \quad \text{in } \Omega \times (0, \infty)$$

$$u(\mathbf{x}, t) = 0 \quad \text{on } \partial\Omega$$

Then by separation of variables, $u = T(t)\phi(\mathbf{x})$, ϕ must satisfy the (PDE) EVP

$$\begin{cases} \nabla^2\phi + \lambda\phi = 0 & \text{in } \Omega \\ \phi = 0 & \text{on } \partial\Omega \end{cases} \tag{3}$$

This is a special case of the multi-dimensional Sturm-Liouville problem, so we expect much of what we mentioned for the one-dimensional case should apply here.

Divergence Theorem: Given the conditions on Ω above, then for any smooth vector function $\mathbf{w} = \mathbf{w}(\mathbf{x})$,

$$\int_\Omega div\ \mathbf{w}\ d\mathbf{x} = \int_{\partial\Omega} \mathbf{w} \cdot \nu\ ds \ . \tag{4}$$

Comment on notation: In these Notes, vectors will generally be in bold Roman letters except if the vectors are of unit length, like the normal vector, ν, above, in which case we will use unbolded Greek letters. The context should make vector versus scalar notation apparent.

If $\mathbf{w} = u\nabla v = u\ grad\ v$, then $div\ \mathbf{w} = div(u\nabla v) = u\nabla^2 v + \nabla u \cdot \nabla v$ (a vector form of the product rule). Then, from the divergence theorem,

$$\int_\Omega u\nabla^2 v d\mathbf{x} + \int_\Omega \nabla u \cdot \nabla v d\mathbf{x} = \int_{\partial\Omega} u\nabla v \cdot \nu\ ds\ . \tag{5}$$

This is a kind of multi-dimensional version of the integration-by-parts formula, and (5) is called **Green's first identity**. Often times $\nu \cdot \nabla v$ is written in directional derivative notation, that is as $\partial v / \partial \nu$; then (5) can be written as

$$\int_{\Omega} u \nabla^2 v \, d\mathbf{x} = \int_{\partial\Omega} u \frac{\partial v}{\partial \nu} ds - \int_{\Omega} \nabla u \cdot \nabla v \, d\mathbf{x} \, . \tag{6}$$

If $u = v = \phi$, where ϕ is an eigenfunction associated with λ in the EVP (3), Then (6) becomes

$$\int_{\Omega} \phi \nabla^2 \phi \, d\mathbf{x} = - \int_{\Omega} |\nabla \phi|^2 \, d\mathbf{x} \ . \tag{7}$$

Multiply the equation in (3) by ϕ and integrate, using (7), we obtain

$$- \int_{\Omega} |\nabla \phi|^2 \, d\mathbf{x} + \lambda \int_{\Omega} \phi^2 \, d\mathbf{x} = 0 \ ,$$

or,

$$\lambda = \frac{\int_{\Omega} |\nabla \phi|^2 \, d\mathbf{x}}{\int_{\Omega} \phi^2 \, d\mathbf{x}} =: \mathcal{R}[\phi] > 0 \tag{8}$$

for the eigenfunction ϕ. The quotient here is the (multi-dimensional) Rayleigh quotient associated with problem (3).

Given the domain above, and problem (3), we have the following minimization result:

Theorem 1: Given $\mathcal{A}$, the set of functions defined on the closure of Ω, $\bar{\Omega}$, continuous on $\bar{\Omega}$, piecewise smooth in Ω, not identically zero in Ω, but zero on $\partial\Omega$, then for any $\psi \in \mathcal{A}$, the smallest eigenvalue, λ_1, of (3), satisfies $\lambda_1 \leq \mathcal{R}[\psi]$, and $\mathcal{R}[\psi] = \lambda_1$ if, and only if ψ is an eigenfunction associated with λ_1.

For a piece of the monotonicity theorem generalized to multi-dimensions, we have

Theorem 2: Given the domain Ω, and subdomain $\tilde{\Omega}$ in Ω, $\tilde{\Omega}$ not equal to Ω, let $\lambda_1(\Omega)$ (respectively, $\lambda_1(\tilde{\Omega})$) be the smallest eigenvalue of (3) defined on domain Ω (respectively, defined on subdomain $\tilde{\Omega}$). Then $\lambda_1(\Omega) < \lambda_1(\tilde{\Omega})$. Moreover, $\lambda_n(\Omega) < \lambda_n(\tilde{\Omega})$ for all $n = 1, 2, \ldots$.

Example: Consider Ω being the square centered at the origin and having side of length 2; that is, $\Omega = \{(x, y) : |x| < 1, |y| < 1\}$. Let $\tilde{\Omega}$ be the inscribed disk of unit radius. Then $\lambda_1(\Omega) = \pi^2/2 \approx 4.9348$ and $\lambda_1(\tilde{\Omega}) \approx 5.7831 > \lambda_1(\Omega)$. (The first eigenvalue of the disk problem comes from the smallest root of the Bessel function of first kind of zeroth order, namely, $J_0(\sqrt{\lambda_{01}}) = 0$. Again, see Appendix F for a discussion of Bessel functions.)

Question: For the heat problem on a plate of unit thickness, among all shapes of equal area, what shape of plate will cool the slowest?

Hint: Of all the shapes of equal area, the disk has the smallest circumference.

Faber-Kahn inequality lemma; For all domains $\Omega \subset \mathbb{R}^2$ of equal area, the disk has the smallest first eigenvalue.

Thus, recall that $u(\mathbf{x}, t) = \sum_n a_n e^{-\lambda_n D t} \phi_n(\mathbf{x})$, and since $0 < \lambda_1 < \lambda_2 < \ldots$ (D = constant diffusivity), the slowest decaying term (the rate-determining term) is the first term, i.e. the one with the smallest eigenvalue. So other domains with larger λ_1 will have terms decaying faster, that is, cooling faster.

Comment on continuity: It is reasonable to expect that if we have two domains Ω_1 and Ω_2 with Ω_1 "sufficiently close to" Ω_2, then $\lambda_1(\Omega_1)$ should be a close approximation to $\lambda_1(\Omega_2)$. The trick is what do we mean by Ω_1 being close to Ω_2. This is actually a nontrivial issue not easily resolved. It turns out there needs to be a really smooth function near the identity function between Ω_1 and Ω_2 for one to make rigorous sense of this, so the conditions needed to have such an approximation are really special.

19.3 Can one hear the shape of a drum?

This is the title of a famous paper by Mark Kac[1]. The problem is the following: imagine a domain Ω (interior of a smooth closed curve in the plane that represents the (arbitrary) shape of a drum head), and suppose we knew all the frequencies of sound exactly that the drumhead can emit. That

[1]American Mathematical Monthly, vol. 73, no. 4, 1966, pp 1-23.

is, we know the precise values of all the eigenvalues $\lambda_n(\Omega)$, $n = 1, 2, 3, \ldots$. Can one infer the shape of Ω from this information?

This is an **inverse problem**; instead of being given the precise description of Ω, and problem (3), and asked to find the eigenvalues, we are given the eigenvalues and asked to find Ω. This is a really hard problem. When Kac proposed the question in 1966 he gave a few preliminary results but could not answer the question. The question actually goes back to 1910, where it was mentioned in a slightly different form by Dutch physicist Lorentz. One of the great mathematicians of the twentieth century, Hermann Weyl, some years later (around 1921) proved a limited result that can be expressed as $N(\lambda) \sim \frac{|\Omega|}{2\pi}\lambda$ as $\lambda \to \infty$, where $N(\lambda)$ = number of eigenvalues $< \lambda$, and $|\Omega|$ = area of Ω. This says as λ gets large, since we know the distribution of the eigenvalues, we also know the function $N(\lambda)$, so that we can "hear the area of the drumhead". Not overly satisfying result considering the original goal. Kac's question was finally answered negatively in 1993 for the case of polygonal domains (boundaries have "corners", that is, origami type domains) when it was proved there were two such unequal domains with exactly the same eigenvalues. For domains with smooth boundaries, the problem remains open to this day.

Inverse problems come up a lot in science, and are important in industry. Often they are phrased as parameter-determining problems. Sometimes they are source problems. For example, from an advection-diffusion equation standpoint, suppose we have a flowing river where he have knowledge of the current. We have placed sensors every so often along the rive to monitor for a certain toxin. All of a sudden we pick up a sensor reading between sensors. From the data can you locate the point source of the toxin release between the two sensors?

19.4 Comments on examples from the previous section

Completion of Example 1: From (2) and (5) in the last section we have

$$f(x) = (1+x)^{-1/2} \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{\ln(2)} \ln(1+x)\right) . \tag{9}$$

Since $\sigma(x) \equiv 1$, multiply both sides of (9) by an arbitrary eigenfunction $\phi_m(x) = \frac{1}{\sqrt{1+x}} \sin(\frac{m\pi \ln(1+x)}{\ln 2})$ and integrate:

$$\int_0^1 f(x)(1+x)^{-1/2}\sin(\frac{m\pi\ln(1+x)}{\ln 2})dx =$$

$$\sum_{n=1}^{\infty} B_n \int_0^1 (1+x)^{-1}\sin(\frac{n\pi\ln(1+x)}{\ln 2})\sin(\frac{m\pi\ln(1+x)}{\ln 2})dx =$$

$$\ln 2 \sum_{n=1}^{\infty} B_n \int_0^1 \sin(n\pi y)\sin(m\pi y)dy$$

$$= \frac{\ln 2}{2} B_m \quad ,$$

where we have defined $y = \ln(1+x)/\ln(2)$ (so $\ln(2)\ dy = (1+x)^{-1}dx$), and used the orthogonality of $\{\sin(n\pi y)\}_{n\geq 1}$ on $[0,1]$. Hence,

$$B_n = \frac{2}{\ln 2}\int_0^1 \frac{f(x)}{\sqrt{1+x}}\sin(\frac{n\pi\ln(1+x)}{\ln 2})\ dx \quad ,$$

with $u(x,t)$ given by (5):

$$u(x,t) = (1+x)^{-1/2}e^{-D_0 t/4}\sum_{n=1}^{\infty} B_n e^{-D_0 n^2\pi^2 t/(\ln 2)^2}\sin(\frac{n\pi\ln(1+x)}{\ln 2}) \quad .$$

Comment: The bottom line here is that we have done the same strategy on this variable coefficient problem as we did with previous examples; transform the original problem, if necessary, to one with homogeneous boundary conditions, separate variables to obtain the problem's EVP. Solve for the eigenvalues - eigenfunction pairs, if possible, then solve for the t-dependent problem. Whether we are talking about diffusion or wave equations, the only unused information is the initial condition(s). So sum the contributions, employing the superposition principle and apply the initial conditions on the resulting eigenfunction series for the solution in order to obtain expressions for the Fourier coefficients. Integrate the expressions, if possible.

Comment on exercise, page 2 of the last section: You should obtain eigenvalues $\lambda_n = 1/4 + (n\pi/\ln(1+l))^2$, and associated eigenfunctions $\phi_n(x) = \sqrt{1+x}\sin(\frac{n\pi\ln(1+x)}{\ln(1+l)})$ to the eigenvalue problem

$$(1+x)^2\frac{d^2\phi}{dx^2} + \lambda\phi = 0 \quad , \quad \phi(0) = \phi(l) = 0 \tag{10}$$

(which is a Cauchy-Euler equation). Hence, the solution $u(x,t)$ for the displacement of the string from equilibrium (which you were not asked to provide) is

$$u(x,t) = \sqrt{1+x}\sum_{n=1}^{\infty}\{a_n\cos(c\sqrt{\lambda_n}t) + b_n\sin(c\sqrt{\lambda_n}t)\}\sin[\frac{n\pi\ln(1+x)}{\ln(1+l)}] \ .$$

Thus,

$$u_t(x,t) = \sqrt{1+x}\sum_{n=1}^{\infty}c\sqrt{\lambda_n}\{-a_n\sin(c\sqrt{\lambda_n}t)+b_n\cos(c\sqrt{\lambda_n}t)\}\sin[\frac{n\pi\ln(1+x)}{\ln(1+l)}] \ ,$$

so

$$u_t(x,0) = 0 = \sqrt{1+x}\sum_{n=1}^{\infty}c\sqrt{\lambda_n}\ b_n\sin[\frac{n\pi\ln(1+x)}{\ln(1+l)}] \ ,$$

which implies $b_n = 0$ for all $n \geq 1$. Also,

$$u(x,0) = f(x) = \sqrt{1+x}\sum_{n=1}^{\infty}a_n\sin[\frac{n\pi\ln(1+x)}{\ln(1+l)}] \ . \tag{11}$$

But note that the EVP (10) is **not** in Sturm-Liouville form! That is, we should write

$$\frac{d^2\phi}{dx^2} + \lambda(1+x)^{-2}\phi = 0 \qquad (\text{so } \sigma(x) = (1+x)^{-2},\ p \equiv 1,\ q \equiv 0) \ ,$$

and compute the coefficients a_n in (11) by multiplying both sides by an arbitrary $\phi_m(x)\sigma(x) = (1+x)^{-3/2}\sin[\frac{m\pi\ln(1+x)}{\ln(1+l)}]$, and integrate:

$$\int_0^l (1+x)^{-3/2} f(x)\sin[\frac{m\pi\ln(1+x)}{\ln(1+l)}]dx$$
$$= \sum_{n=1}^{\infty} a_n \int_0^l (1+x)^{-1}\sin[\frac{n\pi\ln(1+x)}{\ln(1+l)}]\sin[\frac{m\pi\ln(1+x)}{\ln(1+l)}]dx$$
$$= \ln(1+l)\sum_{n=1}^{\infty} a_n \int_0^l \sin(n\pi y)\sin(m\pi y)dy \qquad \text{where } y = \frac{\ln(1+x)}{\ln(1+l)}$$
$$= \frac{\ln(1+l)}{2}a_m \qquad \text{by orthogonality .}$$

To summarize, the solution to this example is

$$u(x,t) = \sqrt{1+x}\sum_{n=1}^{\infty} a_n \cos(c\sqrt{\lambda_n}t)\sin[\frac{n\pi\ln(1+x)}{\ln(1+l)}]$$

where

$$\lambda_n = \frac{1}{4} + \frac{n^2\pi^2}{(\ln(1+l))^2}$$

and

$$a_n = \frac{2}{\ln(1+l)}\int_0^l (1+x)^{-3/2} f(x)\sin[\frac{n\pi\ln(1+x)}{\ln(1+l)}]dx \ .$$